crack. Inverting the order of integration in (4.15), we obtain

$$\gamma = \frac{\sigma_0}{2a}\frac{d}{da}\int_0^a p(t)\,t\,dt$$

It can be shown that

$$\int_0^a p(t)\,t\,dt = \frac{g_1\sigma_0}{g_{22}}\frac{2}{3}\varkappa a^3(1+4\varkappa^2)$$

Then the value of the critical load applied to the crack edges is

$$\sigma_0 = \sqrt{\frac{\gamma g_{21}}{ag\varkappa(1+4\varkappa^2)}}$$

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UDC 62 - 50

ON THE ESCAPE PROBLEM WITH CONSTRAINTS OF DIFFERENT TYPES

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We examine a linear escape problem in which the pursuing player's control is constrained in energy, while that of the escaping player, in absolute value. The game's termination set is defined as the equality of the players' geometric coordinates. We have obtained sufficient conditions for the possibility of evasion from contact from any point of the phase space, not belonging to the game's termination set, and sufficient conditions for the existence of an open set in the phase space, from any point of which the game can be terminated in finite time.

Suppose that in the space R^n $(n \geqslant 2)$ the motion of the pursuing vector x and of the escaping vector y is described by the equations

$$x^{(p)} + a_1 x^{(p-1)} + \ldots + a_{p-1}\, x^{\cdot} = u, \quad x \in R^n \tag{1}$$

$$y^{(q)} + b_1 y^{(q-1)} + \ldots + b_{q-1}\, y^{\cdot} = v, \quad y \in R^n$$

$$\int_0^\infty (u \cdot u)\, ds \leqslant \rho^2, \quad \rho > 0; \quad |v| \leqslant \sigma, \quad \sigma > 0 \tag{2}$$

Here $a_1, \ldots, a_{p-1}, b_1, \ldots, b_{q-1}$ are real numbers, u, v are control vectors chosen in a class of measurable vector-valued functions, satisfying constraints (2). Such controls are called admissible. The pursuit is considered to be ended at an instant t_1 if at this instant the equality $x(t_1) = y(t_1)$ is first satisfied. We say that evasion from contact is possible from the point

$$z_0 = x(0), \ldots, x^{(p-1)}(0), y(0), \ldots, y^{(q-1)}(0) \in R^{n(p+q)}$$

if at each instant t we can select the escaping control $v(t)$ so that the equality $x(t) = y(t)$ is not satisfied for any value of t whatsoever. It is assumed that when forming the control $v(t)$ the escaping knows the vectors $x(t)$, $y(t)$, $z(t)$, Eqs. (1) and constraints (2). In addition, it is assumed that the pursuer's control resource, i.e.

$$\rho^2(t) = \rho^2 - \int_0^t (u \cdot u)\, ds$$

is known at each instant t.

We introduce the following notation: π is the orthogonal projection operator from space R onto any fixed subspace R^2 and $\xi(t) \equiv \pi\Delta(t) \equiv \pi(x(t) - y(t))$. The absolute value of the vector $(x^{\cdot}(t), \ldots, x^{(p-1)}(t), y^{\cdot}(t), \ldots, y^{(q-1)}(t))$ is denoted by $\eta(t)$ and $\chi(t) \equiv (1 - \eta(t))^{-1}$; integration with respect to s from 0 to t is denoted by angle brackets $(\langle g(s)\rangle)$.

T h e o r e m. If $p > q$, then evasion from contact is possible from any point z_0 $(\Delta(0) \neq 0)$; evasion can occur in such a way that the following estimate holds. If $p > q$, there exist three positive constants $\Theta_0, \varepsilon_0, c$, depending only on Eqs. (1) and conditions (2), a nondecreasing sequence ε_k, and a sequence $T_k \to +\infty$, depending on the evasion process

a) $|\Delta(t)| \geqslant \varepsilon_0, \quad t \in [0, T_1]; \quad$ if $\quad |\Delta(0)| > \varepsilon_0$

b) $|\Delta(t)| > c(|\Delta(0)|\, \chi(t))^q, \quad t \in [0, T_1]; \quad$ if $\quad |\Delta(0)| \leqslant \varepsilon_0$

c) $|\Delta(t)| > c(\varepsilon_{k-1}\, \chi(t))^q, \quad t \in [T_k, T_{k+1}]$

If $p \leqslant q$, these exists an open set in space $R^{n(p+q)}$, from any point of which the pursuit can be completed in finite time.

P r o o f. Suppose that at an instant T_k the vector $z(T_k)$ is such that $\Delta(T_k) = 0$, $\rho^2(T_k) > 0$.

1°. For any admissible controls $u(t)$, $v(t) \equiv v_k$ specified on an interval $[0, M]$

$$\xi(T_k + t) = \pi(\varphi_1(t, T_k) - \varphi_2(t, T_k)) - \pi(v_k - h(t))\langle \mu_{q-1}(s)\rangle$$

$$\varphi_1(t, T_k) = \sum_{i=0}^{p-1} \gamma_i(t)\, x^{(i)}(T_k), \quad \varphi_2(t, T_k) = \sum_{i=0}^{q-1} \mu_i(t)\, y^{(i)}(T_k)$$

where $\gamma_i(t)$, $\mu_i(t)$ are solutions of the equations (δ_{il} is the Kronecker symbol)

$$\gamma^{(p)} + a_1\gamma^{(p-1)} + \ldots + a_{p-1}\gamma^{\cdot} = 0, \quad \gamma_i^{(l)}(0) = \delta_{il} \tag{3}$$
$$\mu^{(q)} + b_1\mu^{(q-1)} + \ldots + b_{q-1}\mu^{\cdot} = 0, \quad \mu_i^{(l)}(0) = \delta_{il}$$
$$|h(t)| < c\rho(T_k)\, t^{p-q-1/2} \tag{4}$$

(c depends only on the solutions of Eqs. (3)). The number M is determined from the condition $\langle\mu_{q-1}(s)\rangle > 0$ for $t \in [0, M]$.

Let admissible controls $u(t), v_k$ be given on the interval $[0, M]$; then by the Cauchy formula

$$x(T_k + t) = \varphi_1(t, T_k) + \langle\gamma_{p-1}(s)u(t-s)\rangle$$
$$y(T_k + t) = \varphi_2(t, T_k) + \langle\mu_{q-1}(s)\rangle v_k$$

After projection onto R^2 we obtain the equality

$$\xi(T_k + t) = \varphi(t, T_k) - \pi(v_k - \langle\gamma_{p-1}(s)\, u(t-s)\rangle \langle\mu_{q-1}(s)\rangle^{-1}) \langle\mu_{q-1}(s)\rangle$$
$$\varphi(t, T_k) = \pi(\varphi_1(t, T_k) - \varphi_2(t, T_k))$$

It is obvious that for $t \in [0, M]$

$$|t^q \langle\mu_{q-1}(s)\rangle| < c_1$$

Applying the Cauchy-Buniakowski inequality, we find that inequality (4) is valid for the function

$$h(t) = \langle\gamma_{p-1}(s)u(t-s)\rangle \langle\mu_{q-1}(s)\rangle^{-1}$$

2°. Let Σ be a finite-dimensional linear set of functions analytic on the interval $[0, M]$ and Γ be a square defined by the inequalities $|r^i| \leqslant b$, $i = 1, 2$. Then a positive number ν exists such that for any $\Psi(t) = (\Psi^1(t), \Psi^2(t))$ whose components belong to Σ, we can find a square $\Gamma' \subset \Gamma$ with side 2λ such that the point

$$\xi(T_k + t) = \Psi(t) - \alpha\langle\mu_{q-1}(s)\rangle \tag{5}$$

satisfies the condition $|\xi(T_k + t)| \geqslant \nu t^q$ for $t \in [0, M]$, $\alpha \in \Gamma'$. The proof of this is easily obtained by using the result of section (B) in [1].

3°. Let $|\xi(T_k)| \neq 0$ at some instant T_k ; then there exist a number Θ_k and a control $v(t) \equiv v_k$ such that the inequality

$$|\xi(T_k + t)| > \nu t^q \tag{6}$$

is satisfied for any admissible control $u(t)$ for $t \in [0, \Theta_k]$

Consider the set $Q = \{\pi v, |v| \leqslant \sigma\}$. It is evident that Q is a circle of radius σ with center at the origin of space R^2. In the circle we inscribe a square Γ and in space R^2 we consider the curve (5) where $\Psi(t) \equiv \varphi(t, T)$. Suppose that the square $\Gamma' \subset \Gamma$ has been chosen in accordance with 2° ; then we set

$$\theta_k = \inf\left(\left(\frac{\lambda}{c\rho(T_k)}\right)^{1/(p-q-1/2)}, M\right) \tag{7}$$

while for $t \in [0, \theta_k]$ we set the control $v(t)$ equal to a constant vector $|v_k| < \sigma$ such that πv_k is the center of square Γ'. In this case, for $t \in [0, \theta_k]$

$$\xi(T_k + t) = \varphi(t, T_k) - \alpha(t)\langle\mu_{q-1}(s)\rangle, \quad \alpha(t) = \pi(v_k - h(t))$$

From inequality (4) and equality (7) it follows that $|h(t)| < \lambda$. Therefore, the inclusion $\alpha(t) \in \Gamma'$ is fulfilled for $t \in [0, \theta_k]$. According to 2° this signifies the validity of inequality (6).

4°. From the result in 3° it follows that for any initial value $z(T_k)$, $\Delta(T_k) \neq 0$, we can lead the point $z(t)$ to the position where

$$\Delta(T_k + \theta_k) \geqslant |\xi(T_k + \theta_k)| > \nu\theta_k^q = \varepsilon_k \tag{8}$$

at the instant $T_k + \theta_k$ by applying the special escape control.

5°. There exists a positive constant c_0, depending only on Eqs. (1), such that for any position $z(t)$, $0 < |\Delta(T_k)| \leqslant \varepsilon$, by applying the special control $v(t)$ for $t \in [0, T_k]$, we can so conduct the point $z(T_k + t)$ that the inequality

$$|\Delta(T_k + t)| > c_0 (|\Delta(T_k)| \chi(T_k + t))^q \tag{9}$$

is satisfied. First of all we note that from Eqs. (1), with due regard to the condition $p > q$, follows the existence of constants $\alpha > 0$, $\beta > 0$ depending only on Eqs. (1) and number M, such that the inequalities

$$\chi(T_k + t)^{-1}\chi(T_k) > \alpha, \quad |\Delta(T_k + t) - \Delta(T_k)| < \beta\chi^{-1}(T_k)\, t$$

are satisfied on the interval $[0, M]$. Therefore, for $t < |\Delta(T_k)| \chi(T_k) / 2\beta$

$$|\Delta(T_k + t)| \geqslant |\Delta(T_k)| / 2 \tag{10}$$

Since the escaping employs the special escape control $v(t) \equiv \boldsymbol{v}_k$, inequality (6) is valid and, consequently, the inequality

$$|\Delta(T_k + t)| \geqslant (|\Delta(T_k)| \chi(T_k) / 2\beta)\, \nu \tag{11}$$

is satisfied for $t \geqslant |\Delta(T_k)| \chi(T_k) / 2\beta$. Inequality (9) follows from inequalities (10), (11).

6°. To prove inequalities (a), (b) and (c) of the theorem, we describe the evasion process. Let $|\Delta(0)| > \varepsilon_0 = \nu\theta_0^q$, then up to the instant T_1 when first the equality $|\Delta(T_1)| = \varepsilon_0$ is satisfied, the escaping chooses an arbitrary control; obviously, inequality (a) is satisfied. At the instant T_1 the constraint on the pursuer's control has the form

$$\int_{T_1}^{\infty} (u \cdot u)\, ds \leqslant \rho^2 - \int_0^{T_1} (u \cdot u)\, ds = \rho^2(T_1)$$

We determine θ_1 by formula (7). It is evident that $|\Delta(T_1)| \leqslant \varepsilon_1 = \nu\theta_1^q$; therefore, by applying the special escape control on the interval $[T_1, T_1 + \theta_1]$ we obtain inequality (c), where $k = 1$, for $t \in [T_1, T_1 + \theta_1]$. From inequality (8) it follows that the inequality $|\Delta(T_k + \theta_k)| > \varepsilon_1$ holds at the instant $T_1 + \theta_1$. Therefore, the escaping employs an arbitrary control up to the instant T_2 when first $|\Delta(T_2)| = \varepsilon_1$. Continuing to act in similar fashion, we obtain inequality (c).

If $|\Delta(0)| \leqslant \varepsilon_0$, then for $t \in [0, \theta_0]$ the escaping employs the special escape control which, according to (9), yields inequality (b) for $t \in [0, \theta_0]$ and $|\Delta(\theta_0)| > \varepsilon_0$. Applying an arbitrary control for $t \in (\theta_0, T_1]$, we obtain inequality (b). We act further just as described above.

The proof of the second part of the theorem is based on the results of [2].

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STABILITY OF THE LAGRANGE SOLUTIONS OF THE RESTRICTED THREE-BODY PROBLEM FOR THE CRITICAL RATIO OF THE MASSES

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For two-dimensional and three-dimensional motions we prove the formal stability of the Lagrange solutions of the circular restricted three-body problem with a critical ratio of the masses of the primary bodies.

1. We consider the motion of three material points, mutually attracting one another according to Newton's law. The equations of motion of the problem admit particular solutions corresponding to a motion under which the three bodies form an equilateral triangle rotating in its plane about the center of mass of the three-body system. We examine the stability of these solutions for the case of the circular restricted problem in which the ratio of the masses of the primary bodies is critical.

Let the units of measurement be chosen so that the angular rate of rotation of the primary attracting bodies, the distance between them, the sum of their masses, and the constant attraction are equal to unity. In these units the mass of the smaller of the attracting bodies is equal to μ. We express the Hamiltonian function of the circular restricted three-body problem close to the triangle solution L_4 in a series [1] and we write it in the form

$$H = H_2 + H_3 + H_4 + \ldots \tag{1.1}$$

$$H_2 = \frac{1}{2}(p_1^2 + p_2^2) + q_2 p_1 - q_1 p_2 + \frac{1}{8} q_1^2 - k q_1 q_2 - \frac{5}{8} q_2^2 + \frac{1}{2}(q_3^2 + p_3^2)$$

$$H_3 = \frac{\sqrt{3}}{144}(-28 k q_1^3 + 27 q_1^2 q_2 + 132 k q_1 q_2^2 + 27 q_2^3 - 48 k q_1 q_3^2 - 108 q_2 q_3^2)$$

$$H_4 = \frac{1}{384}(111 q_1^4 + 400 k q_1^3 q_2 - 738 q_1^2 q_2^2 - 720 k q_1 q_2^3 - 9 q_2^4 +$$

$$72 q_1^2 q_3^2 + 960 k q_1 q_2 q_3^2 + 792 q_2^2 q_3^2 - 144 q_3^4), \qquad k = \frac{3\sqrt{3}}{4}(1 - 2\mu)$$

where H_m is a polynomial of degree m in the coordinates q_i and the momenta p_i, $i = 1, 2, 3$.

We consider first the case of two-dimensional motion. The frequencies ω_1 and $\omega_2 (\omega_1 \geqslant \omega_2)$ of an oscillating system with the Hamiltonian $H_2(q_1, q_2, p_1, p_2)$ satisfy the equation

$$\omega_4 - \omega^2 + {}^{27}/_4\, \mu (1 - \mu) = 0$$

To a first approximation we write the stability region in the form of the inequalities

$$0 < \mu < (9 - \sqrt{69}) / 18 \simeq 0.0385208\ldots \tag{1.2}$$